Chapter 7: Exploration in MAB
(Continued)

Recap: MAB

Interactive learning process:

For $t = 0 \rightarrow T - 1$

(# based on historical information)

1. Learner pulls arm $I_t \in \{1,\ldots,K\}$

2. Learner observes an i.i.d reward $r_t \sim \nu_{I_t}$ of arm I_t

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Learning metric:

$$\text{Regret}_T = T\mu^\star - \sum_{t=0}^{T-1} \mu_{I_t}$$

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The Explore and Commit Algorithm:

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For $k = 1 \to K$: (# Exploration phase)

Pull arm-k N times, observe $\{r_i\}_{i=1}^{N} \sim \nu_k$

Calculate arm k's empirical mean: $\hat{\mu}_k = \sum_{i=1}^{N} r_i / N$

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For $t = NK \to T - 1$: (# Exploitation phase)

Pull the best empirical arm, i.e., $I_t = \arg\max_{i \in [K]} \hat{\mu}_i$

Recap: MAB

[Theorem] Fix $\delta \in (0,1)$, set $N = \left(\frac{T\sqrt{\ln(K/\delta)}}{2K}\right)^{2/3}$, with probability at least $1 - \delta$, **Explore and Commit** has the following regret:

$$\text{Regret}_T \leq O\left(T^{2/3}K^{1/3} \cdot \ln^{1/3}(K/\delta)\right)$$

Question for Today:

Can we design an algorithm that achieves $\widetilde{O}(\sqrt{T})$ regret?

Outline:

1. The upper Confidence Bound Algorithm

2. Analysis of UCB algorithm

Statistics that we maintain during learning:

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$$\text{i.e., } \hat{\mu}_t(i) = \sum_{\tau=0}^{t-1} \mathbf{1}\{I_\tau = i\} r_\tau / N_t(i)$$

Recall the Tool for Building Confidence Interval:

[Hoeffding]Given a distribution $\mu \in \Delta([0,1])$, and N i.i.d samples $\{r_i\}_{i=1}^N \sim \mu$, w/ probability at least $1 - \delta$, we have:

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Thus, we know that for all iteration t, we have the for all $i \in [K]$, w/ prob $1-\delta$,

$$|\hat{\mu}_t(i) - \mu_i| \leq \sqrt{\frac{\ln(KT/\delta)}{N_t(i)}}$$

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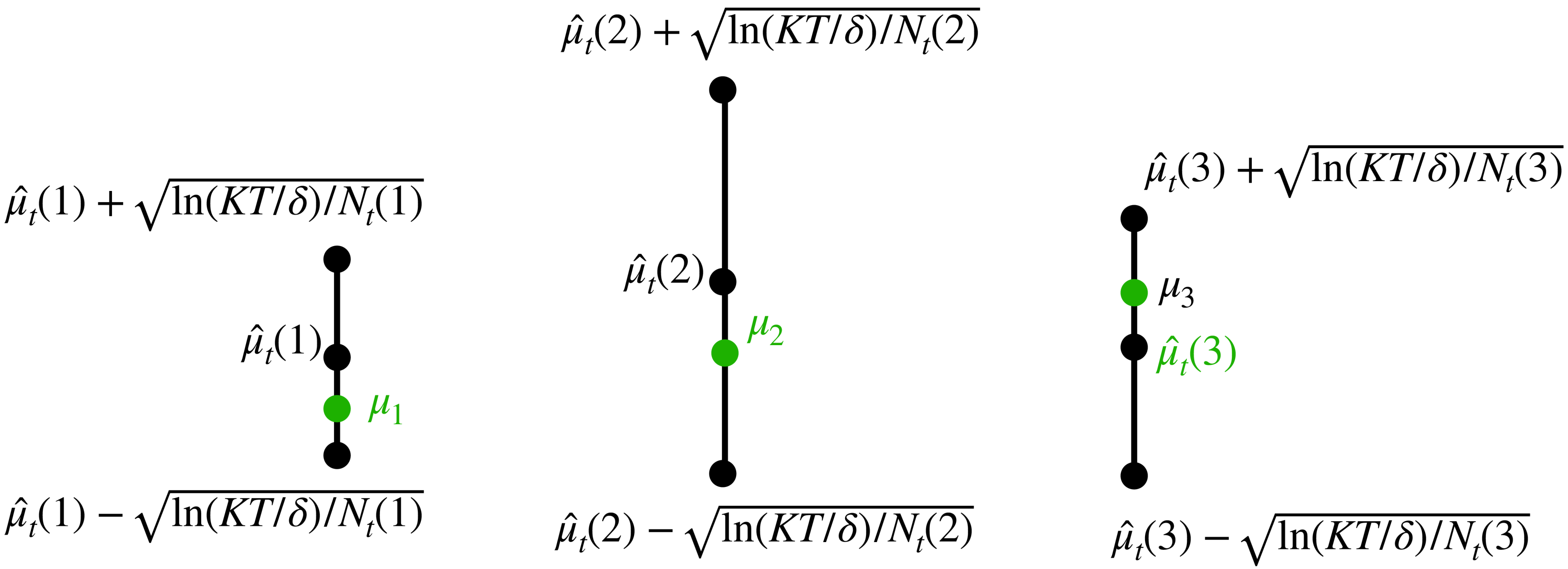

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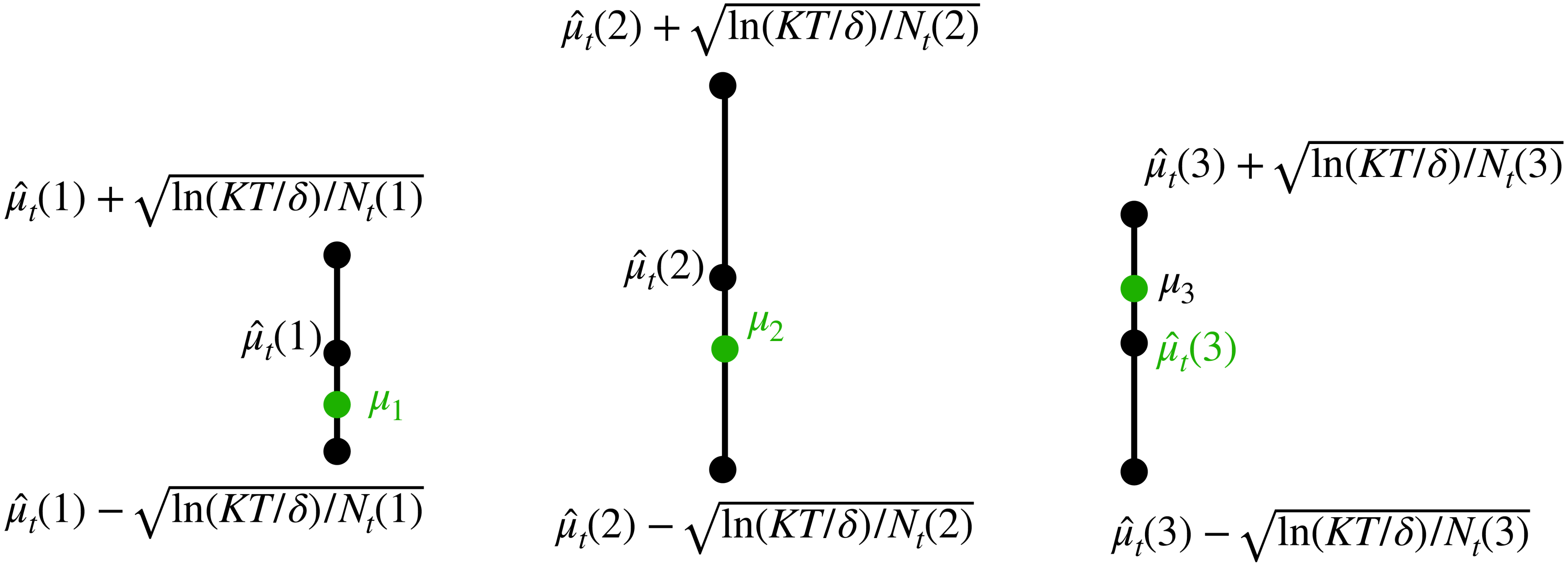

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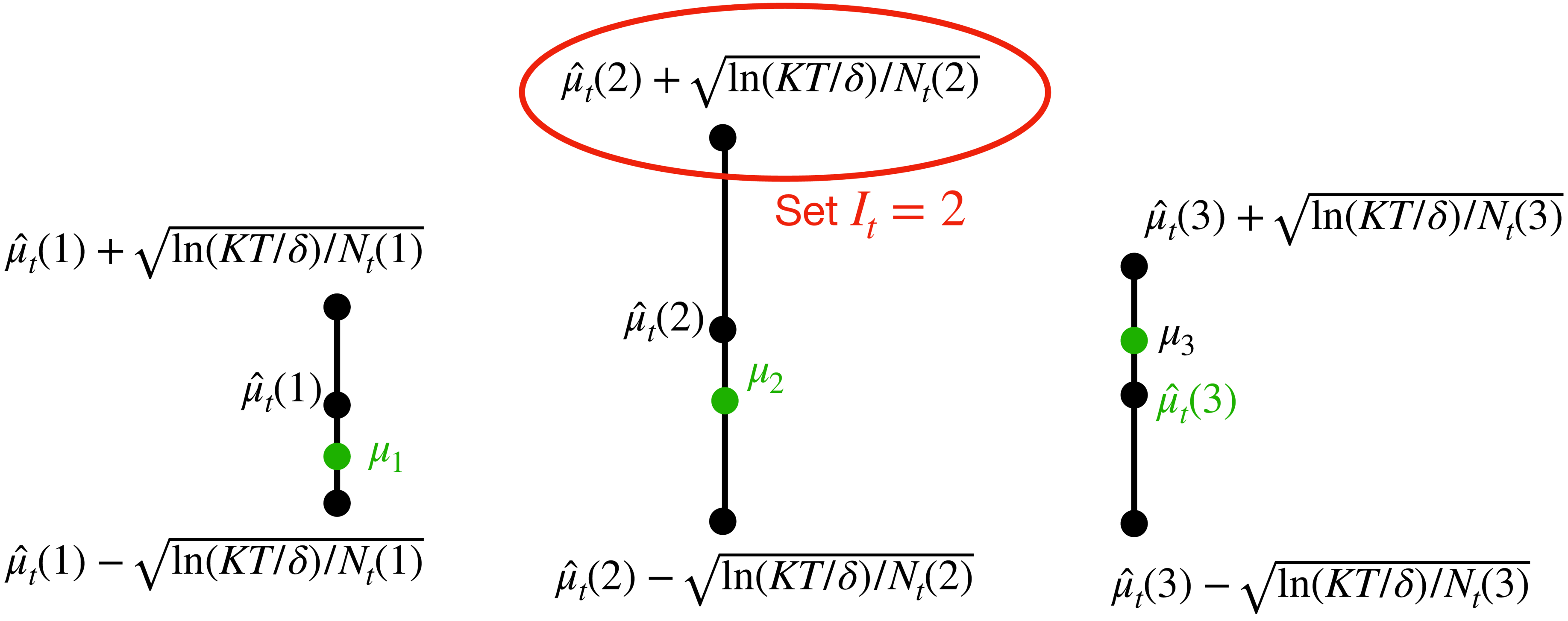

Put things together: UCB Algorithm:

For $t = 0 \rightarrow T - 1$:

$$I_t = \arg\max_{i \in [K]} \left(\hat{\mu}_t(i) + \sqrt{\frac{\ln(KT/\delta)}{N_t(i)}} \right)$$

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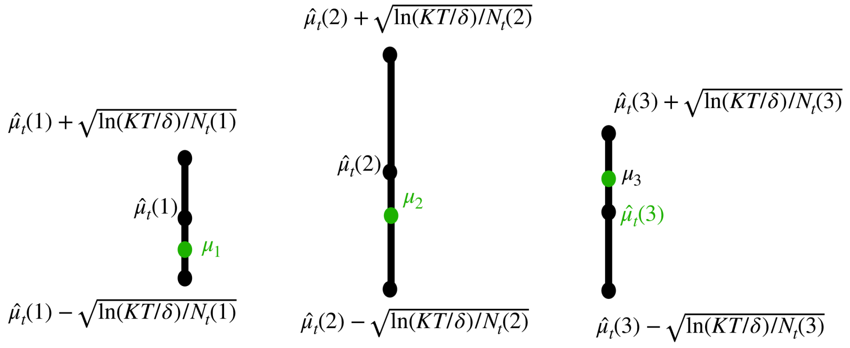

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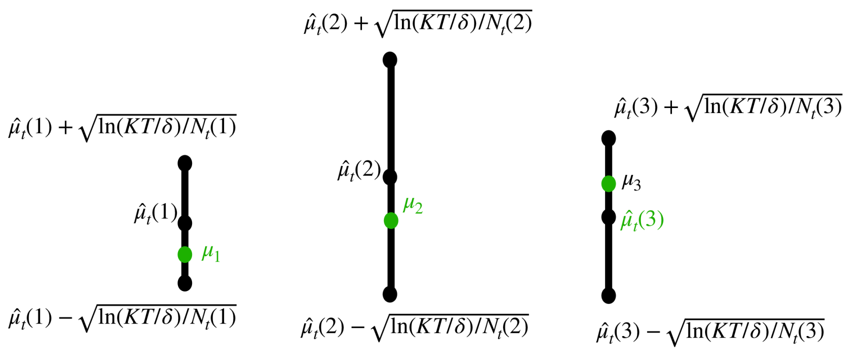

"Reward Bonus": $\sqrt{\frac{\ln(KT/\delta)}{N_t(i)}}$

Outline:

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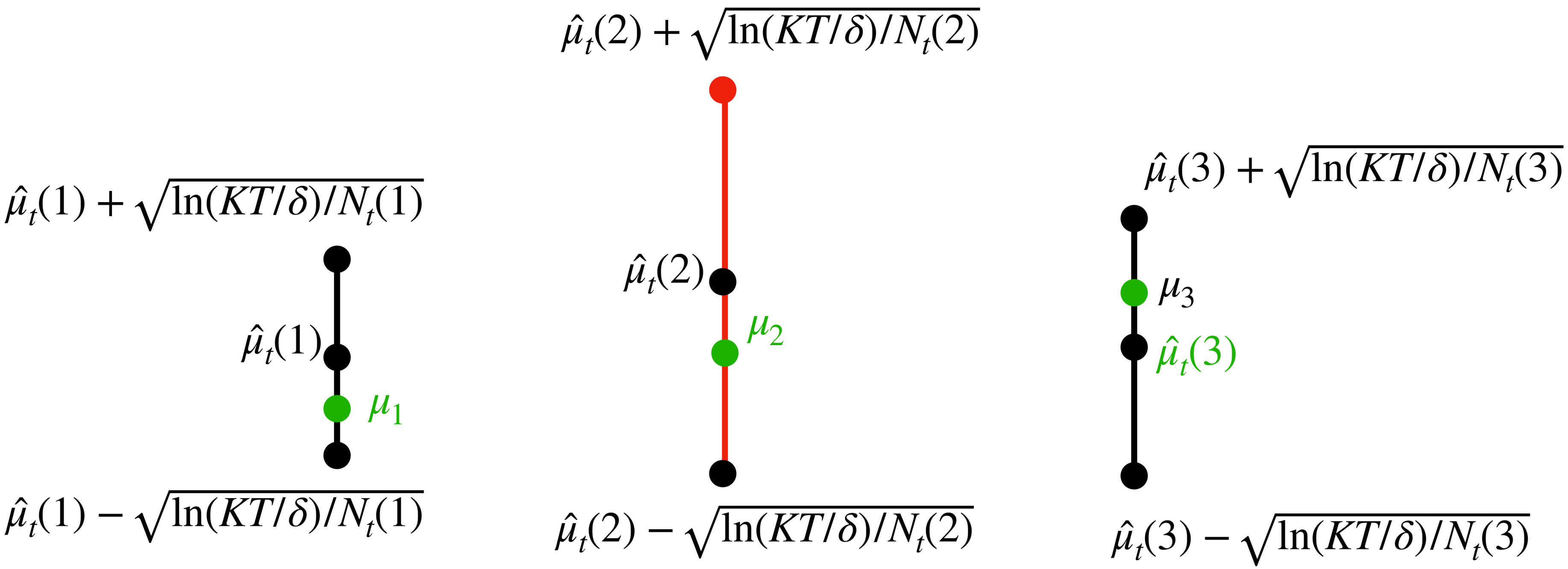
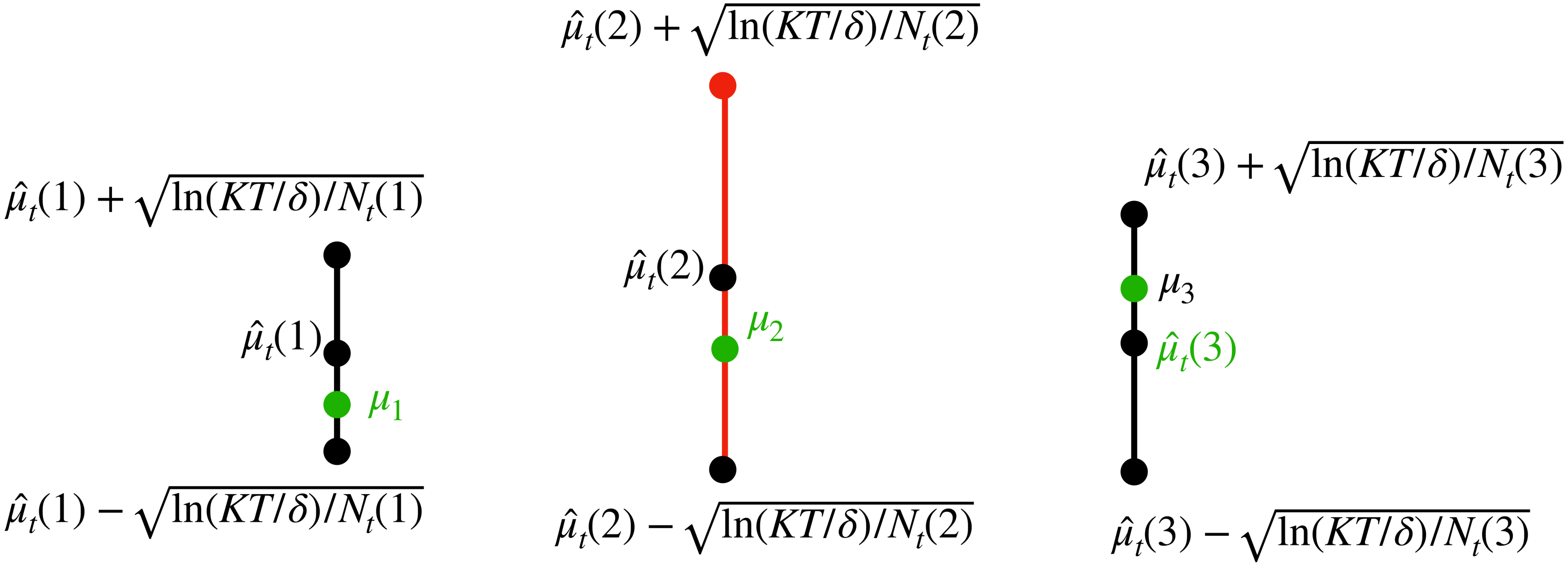


Intuitive Explanation of UCB

Case 1: it has large conf-interval, which means that it has not been tried many times yet (high uncertainty)



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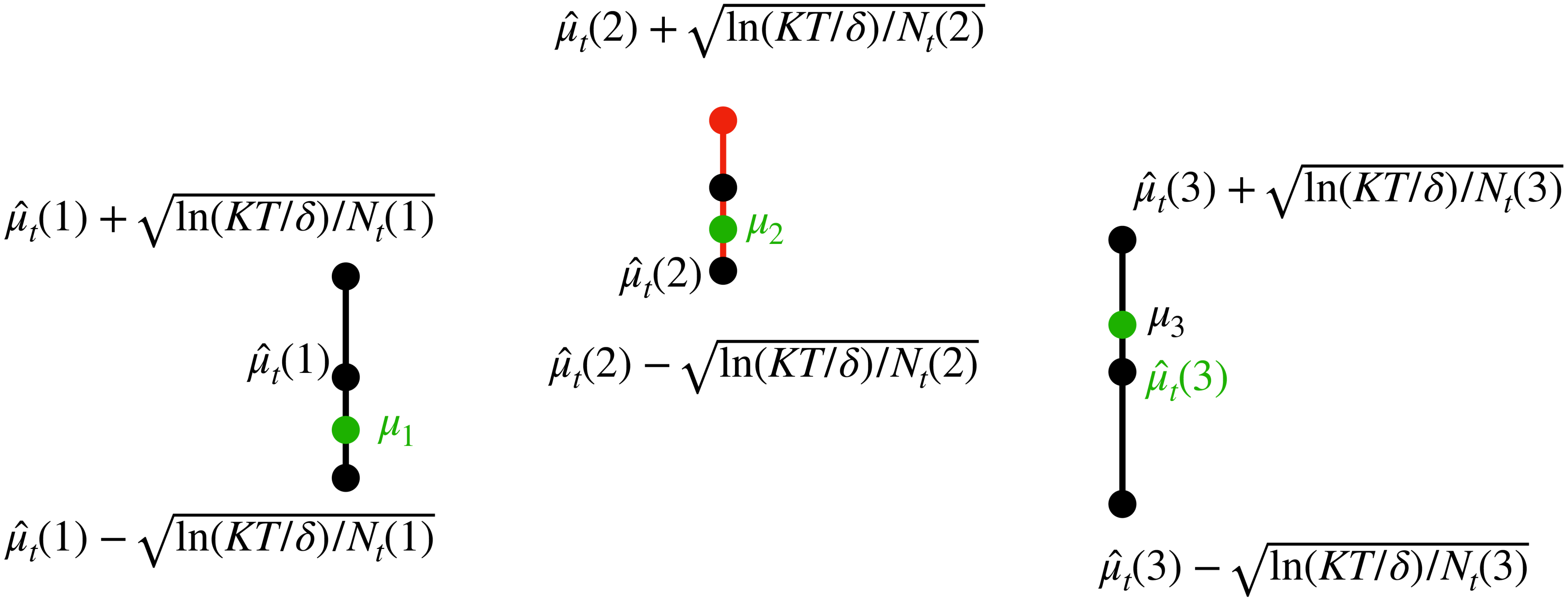
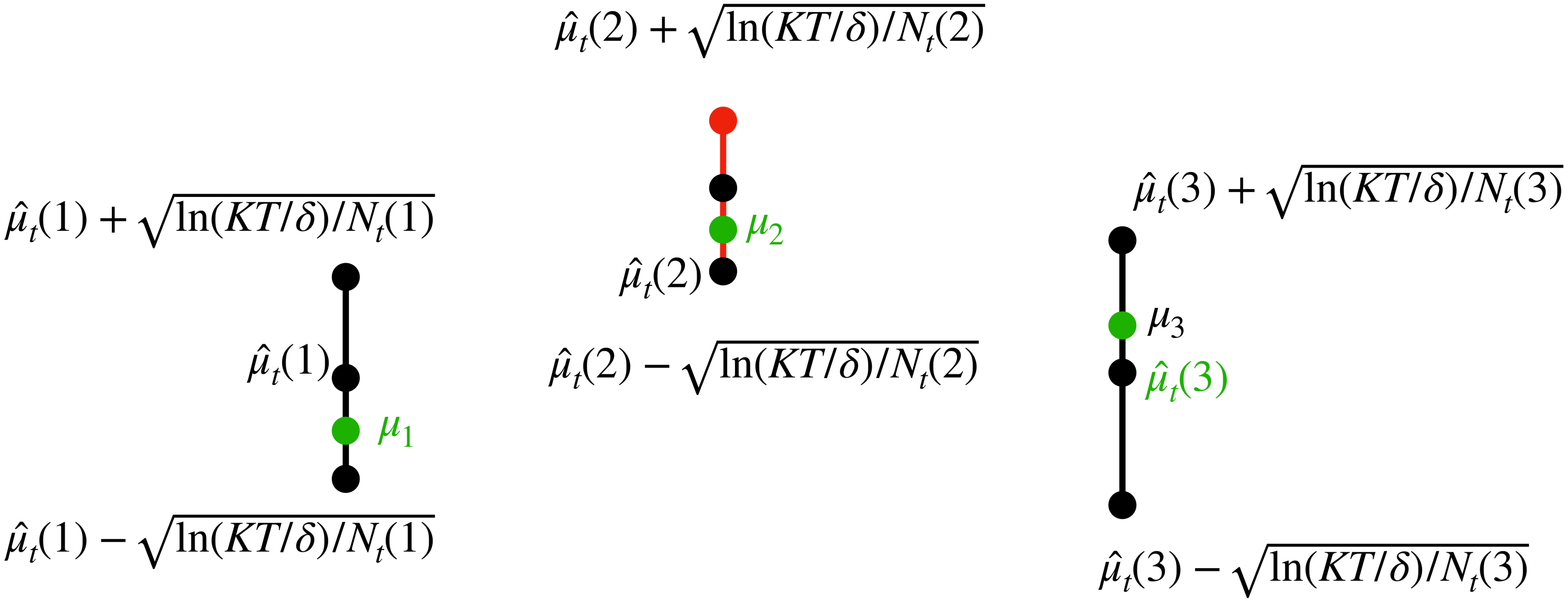


Intuitive Explanation of UCB

Case 2: it has low uncertainty, then it is simply a good arm, i.e., it's true mean is high!



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Thus, we do exploitation in this case!

Let's formalize the intuition

Denote the optimal arm $I^\star = \arg\max_{i \in [K]} \mu_i$; recall $I_t = \arg\max_{i \in [K]} \hat{\mu}_t(i) + \sqrt{\frac{\ln(KT/\delta)}{N_t(i)}}$

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Case 1: $N_t(I_t)$ is small
(i.e., uncertainty about I_t is large);

We pay regret, BUT we **explore** here,
as we just tried I_t at iter t!

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Case 2: $N_t(I_t)$ is large, i.e., conf-interval of I_t is small,

Then we **exploit** here, as I_t is pretty good (the gap between $\mu^\star$ & μ_{I_t} is small)!

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Finally, let's add all per-iter regret together:

$$\text{Regret}_T = \sum_{t=0}^{T-1} \left(\mu^\star - \mu_{I_t}\right)$$

$$\leq \sum_{t=0}^{T-1} 2\sqrt{\frac{\ln(TK/\delta)}{N_t(I_t)}}$$

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Lemma (optional):

$$\sum_{t=0}^{T-1} \sqrt{\frac{1}{N_t(I_t)}} \leq O\left(\sqrt{KT}\right)$$

UCB Regret:

[Theorem (informal)] With high probability, UCB has the following regret:

$$\text{Regret}_T = \widetilde{O}\left(\sqrt{KT}\right)$$

Summary for Today:

UCB algorithm: *Principle of Optimism in the face of Uncertainty*

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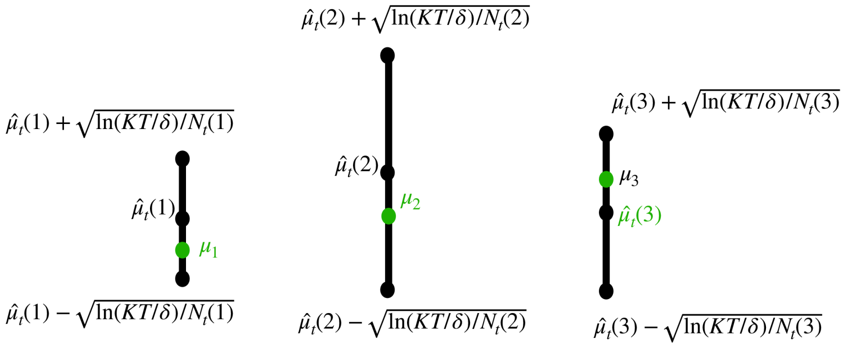

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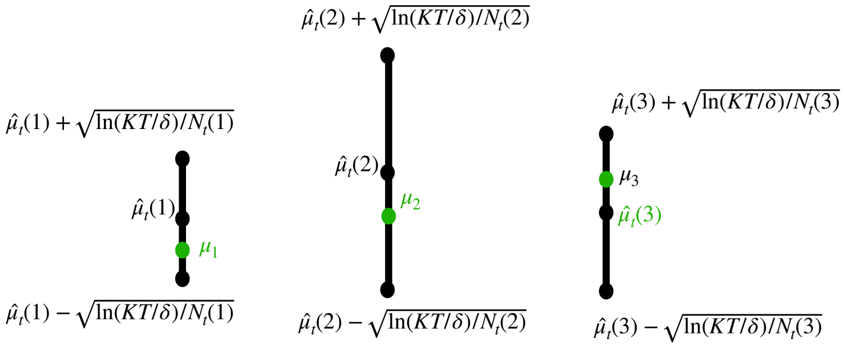

Analysis Intuition:

Case 1: the arm I_t has high uncertainty (we explore)

Case 2: the arm I_t has low uncertainty, then it must be a near-optimal arm (i.e., exploit)

Sample complexity vs. regret

$\alpha \in [0,1)$: T^{α} regret → $\epsilon^{\frac{1}{\alpha-1}}$ sample complexity

$\beta > 0$: $\epsilon^{-\beta}$ sample complexity → $T^{\frac{\beta}{\beta+1}}$ regret